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Amplitude death in the absence of time delays in identical coupled oscillators

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We study the dynamics of oscillators that are mutually coupled via dissimilar (or "conjugate") variables and find that this form of coupling leads to a regime of amplitude death. Analytic estimates are obtained for coupled Landau-Stuart oscillators, and this is supplemented by numerics for this system as well as for coupled Lorenz oscillators. Time delay does not appear to be necessary to cause amplitude death when conjugate variables are employed in coupling identical systems. Coupled chaotic oscillators also show multistability prior to amplitude death, and the basins of the coexisting attractors appear to be riddled. This behavior is quantified: an appropriately defined uncertainty exponent in the coupled Lorenz system is shown to be zero.

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Coupled nonlinear dynamical systems have been extensively studied from both theoretical and experimental points of view. Motivation for such work has come from fundamental as well as practical points of view: systems are rarely isolated and interactions between them frequently give rise to new phenomena which can be exploited in applications. For instance, it has long been known that weak coupling of nonlinear oscillators leads to synchronization [1]. Recent studies have explored the effect of coupling between nonlinear systems that can lead not only to chaotic synchronization but also to a host of other phenomena such as hysteresis, phase locking, phase shifting, phase-flip or amplitude death [1–3]. These studies have broad relevance to many areas of research since nonlinear dynamical systems arise in a variety of contexts in the physical, biological, and social sciences.

In amplitude death (AD) [4] the interaction between two oscillators causes a pair of fixed points to become stable and attracting. These fixed points can either be those which exist (and are unstable) in the uncoupled system, or these can be entirely new fixed points, created by the coupling. AD can be of considerable importance in controlling oscillatory dynamics, and is known to occur when the oscillators are mismatched [5–7] or when the interaction between the oscillators is time delayed [8–11].

In the present work, we study oscillators coupled through dissimilar (or *conjugate*) variables. Our main result is that through this strategy, it is possible to control them to a regime of AD even when the oscillators are *identical*. AD occurs here in the absence of time delay, and thus this is a new scenario for the occurrence of AD [9–13].

Coupling via conjugate variables is natural in a variety of experimental situations where subsystems are coupled by feeding the output of one into the other. An example from the recent literature is provided by the experiments of Kim and Roy on coupled semiconductor laser systems [14], where the photon intensity fluctuation from one laser is used to modulate the injection current of the other, and vice versa.

Coupling through conjugate variables thus appears to provide something such as time-delayed interaction, at least insofar as causing the coupled oscillator system to make a transition to a regime of AD. The similarity between time-delayed variables and conjugate variables has been extensively employed since the early 1980s, in the process of attractor reconstruction [15]. As is well known, time series measurements of a single observable can be used to reconstruct an attractor by using delay variables; indeed Takens' embedding theorem [16] asserts that the topological properties of the reconstructed system match those of the true system with appropriate choices of embedding dimension and time delay.

We first present results for the Landau-Stuart oscillators [9,11,12] which are specified by the equation of motion

$$\dot{z} = (1 + i\omega - |z|^2)z. \tag{1}$$

Here ω is the frequency, and $z(t)=x(t)+iy(t)$. When two such oscillators are coupled through conjugate variables (which we take to be dimensionless) the dynamical equations, expressed in Cartesian coordinates are

$$\dot{x}_i = P_i x_i - \omega_i y_i,$$

$$\dot{y}_i = P_i y_i + \omega_i x_i + \epsilon x_j. \tag{2}$$

We have used the notation $P_i = 1-|z_i|^2$, $i,j=1,2$, and $j \neq i$. The parameter ϵ governs the strength of the coupling and for simplicity we take $\omega_i = \omega$.

There are two sets of fixed points for the system: the origin (0,0,0,0) which exists for all ϵ, and the pair $(x_{1*}, y_{1*}, -x_{1*}, -y_{1*})$ given by

$$x_{1*} = \pm \sqrt{\omega/\epsilon(1+\sqrt{\epsilon\omega-\omega^2})}$$

and

$$y_{1*} = \pm \sqrt{(1-x^2)+\sqrt{\epsilon\omega-\omega^2}}, \tag{3}$$

which exist only for nonzero ϵ. For $\epsilon > \omega$ these are also real and stable; thus $\epsilon > \omega$ signifies the region of AD. Eigenvalues of this latter fixed point are computed for the case $\omega = 2$ and shown in Fig. 1(a), along with the largest Lyapunov exponent as a function of the coupling. Prior to $\epsilon=2$ the largest Lyapunov exponent is zero and the second largest (not shown) is negative, indicative of limit cycle behavior. Beyond $\epsilon=2$, there is a region of multistability (boxed) where both the fixed point attractor and limit cycle coexist. On further increasing ϵ, in the AD region, transient trajectories decay monotonically to the fixed points given by Eq. (3) as shown in Fig. 1(b).

Coupling with dissimilar variables can lead to AD even

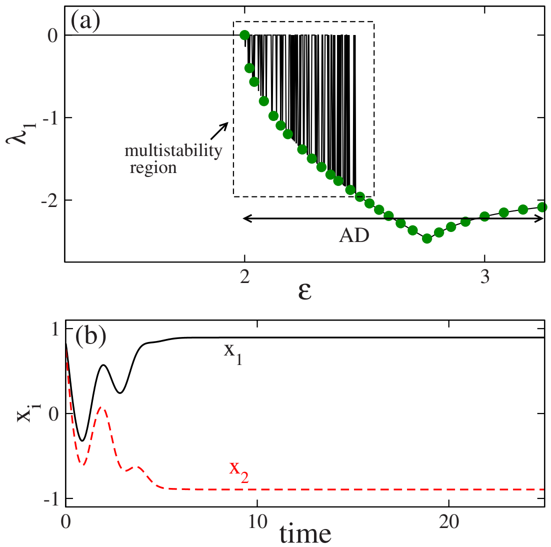

FIG. 1. (Color online) (a) Plot of largest Lyapunov exponent as a function of coupling parameter, ϵ. Circles ($\circ$) correspond to the analytical prediction for the real part of the eigenvalue. The boxed region of the curve as labeled is the region where the fixed points and the limit cycle coexist. Only stable attractors beyond this region are fixed points given by Eq. (3). (b) Transient trajectories of the x components of the two oscillators are shown as a function of time for amplitude death at ϵ=2.5.

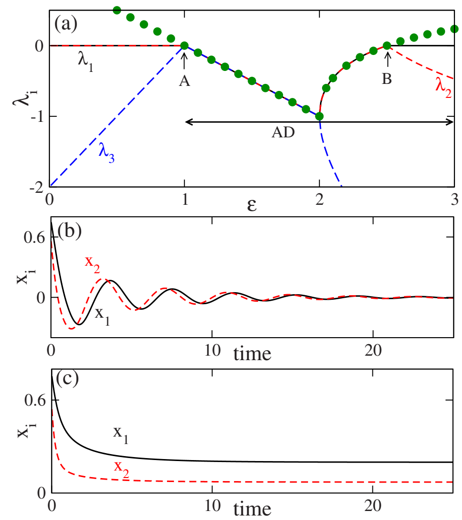

FIG. 2. (Color online) (a) The largest three Lyapunov exponents as a function of coupling parameter, ϵ. Circles ($\circ$) correspond to the analytical prediction for the real part of the eigenvalue. Amplitude death occurs beyond the point marked A. The x components of the two oscillators are shown. AD can occur (b) with oscillation, $\epsilon=1.25$ or (c) a monotonic decrease in the amplitude, at $\epsilon=2.75$.

when the coupling is bidirectional and "diffusive" as in the following Landau-Stuart oscillator system [9,11,12]:

$$\dot{x}_i = P_i x_i - \omega_i y_i + \epsilon(y_j - x_i),$$

$$\dot{y}_i = P_i y_i + \omega_i x_i + \epsilon(x_j - y_i). \tag{4}$$

Now apart from the fixed point at $\mathbf{z}=0$, there are an infinite number of other fixed points, given by the condition

$$x_{1\star}^2 + y_{1\star}^2 = x_{2\star}^2 + y_{2\star}^2 = \frac{\epsilon(y_{1\star}y_{2\star} - x_{1\star}x_{2\star})}{\omega}$$

$$= 1 - \epsilon\left[1 - \left(\frac{x_1 y_1 + x_2 y_2}{x_1 x_2 + y_1 y_2}\right)\right]. \tag{5}$$

At the origin the characteristic eigenvalue equation for (4) is

$$[(1-\epsilon-\lambda)^2 + \omega^2 - \epsilon^2]^2 = 0. \tag{6}$$

Taking $\lambda=\alpha+i\beta$ and separating real and imaginary parts leads to the pair of conditions $\alpha=1-\epsilon$, $\beta=\pm\sqrt{\omega^2-\epsilon^2}$ for $\epsilon<\omega$ and $\alpha=1-\epsilon\pm\sqrt{\epsilon^2-\omega^2}$, $\beta=0$ for $\epsilon>\omega$.

AD now occurs in the interval $1<\epsilon<(1+\omega^2)/2$. The values of ϵ for which $\alpha=0$ are easily determined; thus for $\epsilon<\omega$ there is AD for $\epsilon>1$. Similarly if $\epsilon>\omega$ then there is AD for $\epsilon<(1+\omega^2)/2$. This region can be extended by increasing ω. In addition the inequality $(1+\omega^2)/2>1$ makes AD impossible for frequencies below $\omega=1$.

For purposes of computation, we used $\omega_{1,2}=\omega=2$, as in the previous case. Analytic estimates of the eigenvalue α are shown (with circles) in Fig. 2(a) and compared with numerical calculations of Lyapunov exponent for Eq. (4). All Lyapunov exponents are negative, confirming the occurrence of AD [see Figs. 2(a) and 2(b)] and α, the real part of the eigenvalue at $\mathbf{z}=\mathbf{0}$ coincides with the largest Lyapunov exponent. For $\epsilon<1$ the dynamics are periodic; the only fixed point, $\mathbf{z}=\mathbf{0}$ is unstable, and at ϵ_c, a reverse Hopf bifurcation leads from limit cycle to AD [see Fig. 2(a)]. Here we find that even in the absence of explicit time delay in the coupling, AD occurs over a wide range. On increasing ϵ above $(1+\omega^2)/2$ (beyond the point marked B in Fig. 2), all Lyapunov exponents remain negative, however with the largest Lyapunov exponent being nearly zero. In this regime AD also occurs on another set of fixed points which satisfy the relation, Eq. (5) [17], since the origin is unstable. As the imaginary part of the eigenvalue of the stable fixed point is zero, AD occurs in an overdamped manner, without oscillation; Fig. 2(c). The transition at B appears to be a higher dimensional pitchfork bifurcation [17].

Some comments will serve to put the present results in perspective. Earlier studies of coupled Landau-Stuart oscillators by Aronson *et al.* [6] have shown that AD is only possible for *instantaneous* coupling for $\epsilon>1$ when the intrinsic frequencies are disparate, $|\omega_1-\omega_2|>2\sqrt{2\epsilon-1}$. On the other hand, when there is time delay, AD occurs even for identical oscillators with $\epsilon>1$ [11].

We have studied a number of other systems and find quite generally that coupling via dissimilar variables appears to lead to AD. Numerical results are presented for the system of coupled chaotic Lorenz oscillators, described by the equations

$$\dot{x}_i = 10(y_i - x_i),$$

$$\dot{y}_i = -x_i z_i + 28x_i - y_i + \epsilon(x_j - y_i),$$

$$\dot{z}_i = x_i y_i - \tfrac{8}{3}z_i, \quad i,j=1,2,\ i\neq j. \tag{7}$$

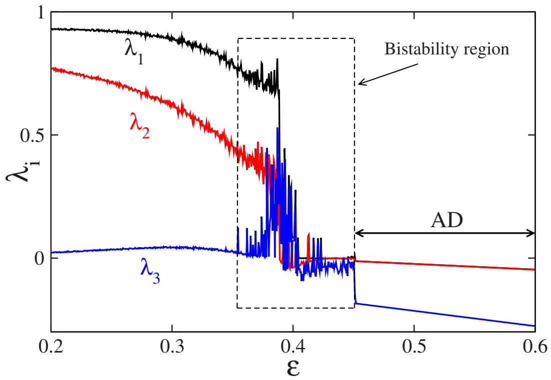

FIG. 3. (Color online) The largest three Lyapunov exponents as a function of the coupling parameter ϵ for the coupled Lorenz oscillators, Eq. (7). The plot shows averaged Lyapunov exponent values for 10 different initial conditions.

This is a six-dimensional dynamical system and the largest Lyapunov exponents are shown in Fig. 3. Immediately preceding the regime of amplitude death—when all Lyapunov exponents become nonpositive—the largest Lyapunov exponents show wild fluctuations as a function of ϵ. There appear to be *two* coexisting attractors with different relative phase relations depending on initial conditions. These attractors are both chaotic, but one is in-phase while in the other, the chaotic oscillators have a mixed-phase relationship as can be seen in Figs. 4(a) and 4(b).

Multistability [18], namely the coexistence of dynamical attractors has also been associated with the phenomenon of *riddling*: the basins of different attractors can be interwoven in a complex manner everywhere. Riddled basins have been

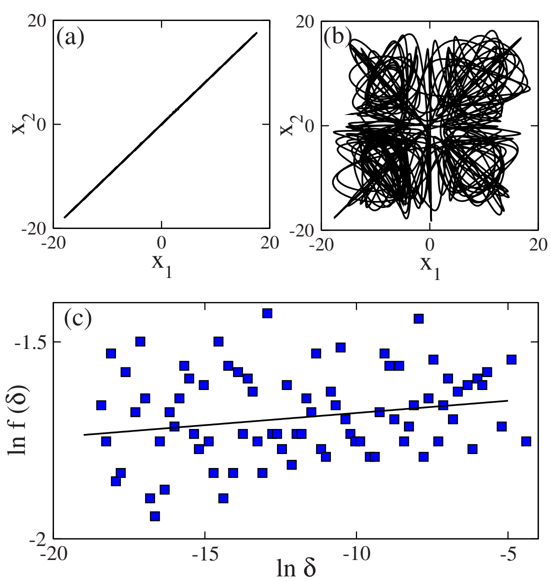

FIG. 4. (Color online) The relative phase between oscillators of coexisting chaotic attractors in the x_1-x_2 plane at $\epsilon=0.38$ for different initial conditions showing (a) chaotic in-phase dynamics and (b) chaotic motion without a specific phase relationship. (c) shows the fraction of uncertain parameter pairs (out of a sample of 200) as a function of the parameter perturbation δ for Lorenz coupled oscillators. This calculation is done for the region of bistability in Fig. 3.

observed in mismatched coupled oscillator systems [18] also for the case of time delays [19]. One consequence is that vanishingly small changes in initial conditions lead to different attractors, making the system completely unpredictable. Dynamics in the uncoupled system ($\epsilon=0$) of identical Lorenz oscillators is chaotic for the parameter values considered. For nonzero ϵ the dynamics changes drastically: the system is bistable for a range of coupling and this eventually leads to AD, as can be seen from the largest three Lyapunov exponents for the coupled system, Fig. 3. All Lyapunov exponents are negative above $\epsilon=0.44$.

Examination of the corresponding basins [17] suggests that these are riddled [17,18] (data not shown here). It is possible, however, to verify the riddling via the computation of an uncertainty exponent [20]. Fixing a perturbation δ and randomly choosing a pair of systems with $|\epsilon-\epsilon'|<\delta$ within the region R, the parameters are termed uncertain if they yield different asymptotic states. The fraction of uncertain parameter pairs $f(\delta)$ typically decreases with δ as a power law,

$$f(\delta) \sim \delta^{\mu}, \tag{8}$$

which defines the uncertainty exponent μ. Results are shown in Fig. 4(c) for the Lorenz system: the exponent is approximately zero (the best fit to the data yields $\mu=0.006\pm0.003$) which is typical of riddle or riddlelike basins [18]. The practical implication is that in this region, the attractor cannot be predicted no matter how small the uncertainty in the specification of parameters.

In summary, in this paper we have studied the effect of coupling systems via conjugate or dissimilar variables. This type of interaction, which frequently arises in experiment, gives rise to phenomena: amplitude death in the absence of delays, and riddling can occur in *identical* coupled systems. Our results appear to apply generally to coupled nonlinear dynamical systems, and have been verified both analytically for coupled Landau-Stuart oscillators and numerically for coupled chaotic oscillators.

We have studied both diffusive and nondiffusive coupling. In cases of AD when the coupling is diffusive, the interaction term can vanish. The stable fixed points are those which are stationary points in the uncoupled subsystems. This applies in the Landau-Stuart case when $x_{i\star}=y_{i\star}=0$ in the region between A and B in Fig. 2(a). It is also possible that the coupling term does not vanish and there are new solutions that are created: this happens in the AD region beyond B where the new fixed points are given by Eq. (5).

In addition to the cases discussed here, we have observed AD and riddling in the cases of mismatched conjugate coupled and time-delay coupled chaotic oscillators and in mismatched conjugate coupled and time-delay coupled limit cycle oscillators. Apart from simple two oscillator systems, we find regimes of oscillator death in the case of a globally coupled network of identical Landau-Stuart oscillators [17], which further underscores the generality of the phenomenon.

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